Filtered Points and the Continued Quest for Theoretical Generality:

A Comparative Case Study

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December 17, 2019

MUSI 718: Mathematical Models of Tonal Systems

ABSTRACT: In tangential response to concerns voiced by Dmitri Tymoczko (2013), I propose a revision to the Filtered Point-Symmetry (FiPS) algorithm that would weaken the symmetry constraint and thus divorce the act of quantization from the concept of maximal evenness. I construct an asymmetric Filtered Point (FiP) configuration using this new algorithm and compare its analytic yield to that of the Tonnetz and the $3 \rightarrow 7 \rightarrow 7 \rightarrow 12 \rightarrow 12$ FiPS configuration, respectively, using Chopin's Prelude Op. 28 No. 9 in E major as a case study. I find that my new configuration suggests a unique hearing of the passage.

I. Background and Preliminaries

A music theorist's first instinct, given a passage of 19th or 20th-century music, may well be to analyze the pitch-classes involved. Often these pitch-classes belong to some well-known *set* or other—a whole-tone or octatonic collection, or some diatonic or pentatonic scale, or some modulatory combination thereof. And if this turns out to be the case and the music does fit some collection or sequence of collections, then the music theorist might feel some satisfaction at being able to *categorize* that passage, to set it in contrast to some other work which uses some other sequence of pitch-class sets.

It is within this narrative of manufactured contrast that Clough and Douthett (1991) attempt to "shift the usual perspective and attend to *similarities*" among these collections using the concept of Maximally Even (ME) sets.[1] ME sets have multiple equivalent definitions.[2] The most relevant definition to this paper, known as the J-function, is as follows:

Suppose we split the octave into *c* equal parts, which we will call *chromatic* pitch classes. We are given some number *d* of *diatonic* notes to arrange in our chromatic array. We intuit that this ordeal may need a bit of counting, so let's number our chromatic notes from 0 to *c–1*. And, to start, let's place some diatonic note on the chromatic note that has index *m*. Now we can algorithmically generate the *k*th note of our maximally even arrangement with the following function:

$$J^{m}_{c,\,d}\,(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor$$

In essence, the J-function simply quantizes one set to one another. Ignoring *m* for a second, we take the following two sets:

$$D = \left\{\frac{0}{d}, \frac{1}{d}, \ \dots\ , \ \frac{(d-1)}{d}\right\} \text{ and } C = \left\{\frac{0}{c}, \frac{1}{c}, \ \dots\ , \ \frac{c-1}{c}\right\}$$

and round each member of *D* down to the nearest member of *C*.[3]

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[1] Clough and Douthett (1991) "Maximally Even Sets" *Journal of Music Theory* Vol. 35.1 93

[2] *Ibid* 96; consider as well the statistical definition, in which the ME set is that in which the variance of all distances is optimized to a minimum.

[3] Bringing *m* back to the equation would mean adding a factor of 1/gcd(c,d) to each member of *D*.

This function yields some familiar yet non-obvious results. The maximally even arrangement of seven notes across twelve pitch classes (which we notate $7 \rightarrow 12$) is the diatonic collection! $5 \rightarrow 12$ yields the pentatonic collection; and $8 \rightarrow 12$ yields the octatonic collection.

But to really appreciate the strength of these results, we must make three more conceptual leaps.

First: Clough and Douthett define an nth-order ME set as an iterated J-function with the following form:

$$J^{m_1, m_2, \dots, m_n}_{d_0, d_1, \dots, d_n}(k) = J^{m_1}_{d_0, d_1}(J^{m_2}_{d_1, d_2}(\dots \ (k)))$$

So now we can look at something like $3 \rightarrow 7 \rightarrow 12$, which turns out to be a major triad when all the circles are aligned at $m_x = 0$.[4]

Second: while we're on the topic, let's take a closer look at *m*. Every integer value of m_x increases every element of D by $\frac{1}{gcf(d_{x-1}, d_x)}$, which is a fancy way of notating the minimum amount needed to change the output of our function.[5] If we treat *m* as a continuous increasing variable, we can cycle through every possible ME set for a given configuration. $7 \rightarrow 12$ now gives us not only the diatonic collection, but a cycle of diatonic collections moving along the circle of fifths![6]

Third: as with most intuitive mathematical concepts, there exists quite a robust visual representation that manages to package all the information we've encoded in the J-function into a much more palatable form. I have reproduced below an example from Plotkin (2019).[7]

We refer to these geometric forms of the J-Function as Filtered Point-Symmetry (FiPS) configurations. We see that *c* is the number of points arranged on the outermost circle; the various iterated

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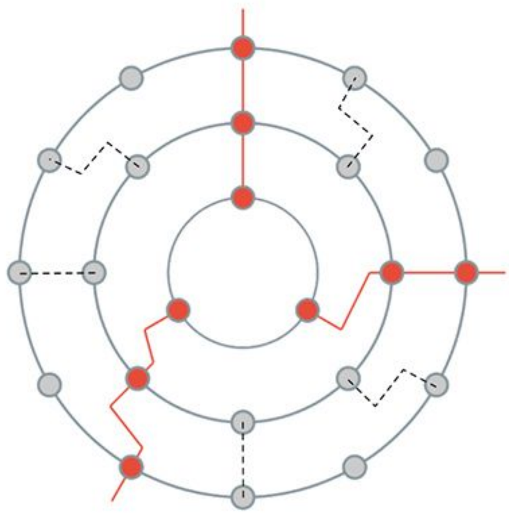
[4] Here, *x* is a placeholder variable for the subscript corresponding to whichever circle we're looking at.
[5] "gcf" indicates greatest common factor.
[6] Readers familiar with Julian Hook's signature transformations might notice a structural similarity—indeed, the similarity is nontrivial—the signature transformation functions are embedded in rotations of the 7-filter. This is explored in depth in Plotkin and Douthett (2013), "Scalar context in musical models" *Journal of Mathematics and Music* Vol. 7.2 106.
[7] Plotkin (2019), "Chord Proximity, Parsimony, and Analysis with Filtered Point-Symmetry" *Music Theory Online* Vol. 25.2

values of *d* correspond to the number of points on the other circles. Incrementing *m* on some layer corresponds to turning that circle clockwise (the direction is chosen arbitrarily). A point on some circle shoots a beam into the nearest point counterclockwise (i.e. opposite the direction of rotation) on the next circle, and so on, until the beam reaches some point on the outermost circle, whose index is output as a pitch-class.

Fig. 1: a $3 \rightarrow 7 \rightarrow 12$ *FiPS configuration. Reproduced from Plotkin (2019)*



The appeal of FiPS is evident: it is simple to construct, and modular by nature. The machine's kinematics are continuous and thus define and ensure a certain measure of parsimony in voice-leading from one output to the next.[8] And crucially, the S of FiPS implies that the near-symmetric sets of notes so common in Western performance practice are derivable from a deeper symmetry. If we buy into these lines of reasoning, then we might believe that Clough and Douthett really did find a unifying mechanism in ME sets, one that underlies the wild world of late-Romantic tonality.

[8] See Plotkin and Douthett (2013) "Scalar context in musical models" *Journal of Mathematics and Music* Vol. 7.2 111-112 for a much more careful discussion of parsimony.

But beautiful, convenient results should always arouse suspicion. FiPS is but one contender among many 'unifying theories' of harmonic or tonal space. Tymockzo (2013) outlines conceptual limitations to FiPS as opposed to Callender/Quinn/Tymoczko's (2008) voice-leading/OPTIC spaces:[9]

1) Scalar context is ill-defined in FiPS.[10] For example, FiPS cannot model major triads existing directly in chromatic space, but must introduce an intermediate 7-division circle that does not necessarily have analytical significance.
2) FiPS cannot easily model scales that are not ME sets, even if they are commonly used alongside ME sets—e.g. the acoustic scale.
3) FiPS cannot model certain voice crossings and runs into issues modeling complex voice leadings.
4) Any FiPS configuration can be retrofit into a path along Tymocko's voice-leading spaces, but not the other way around—this suggests a relative lack of generality.
5) FiPS, being an exercise in abstract algebra, likely holds less conceptual clout in a composer's mind than Tymoczko's Cartesian voice-leading spaces, which themselves attempt to equate distance with voice-leading efficiency—a salient and oft-considered compositional impulse.
6) The J-function only rounds down, not to the nearest point on the next circle. Tymoczko claims this "has important theoretical implications, effectively obscuring the conceptual meaning of the entire approach."[11]

It is not the aim of this paper to address Tymoczko's concerns (though they are worth addressing in detail). Rather, I wish to uncover and question some of the assumptions embedded into FiPS and in

[9] Tymoczko (2013) "Geometry and the quest for theoretical generality" *Journal of Mathematics and Music* Vol. 7.2 139
Callender et. al (2008) "Generalized Voice-Leading Spaces" *Science* Vol. 320.346
[10] I should amend: scalar context *can be* ill-defined in FiPS.
[11] Tymoczko (2013) "Geometry and the quest for theoretical generality" *Journal of Mathematics and Music* Vol. 7.2 140

answering these questions to theorize a more general form of the J-function. In doing so we will find that many of Tymockzo's concerns resolve themselves quite nicely. I contend that this theoretical exercise neither proves nor disproves the non-triviality of FiPS in relation to other spaces; it reinforces the fact that, like any other tool, its analytical quality derives from the actions of the user.

II. Fun with Circles

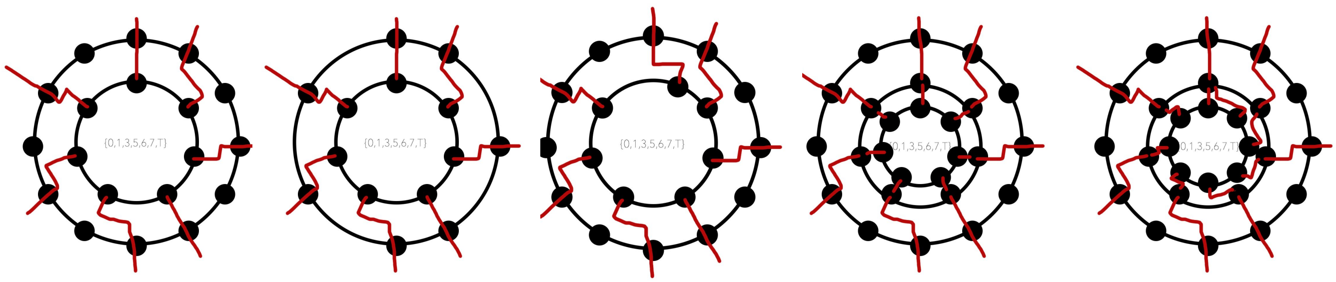
FiPS operates under three unspoken assumptions beside its formal rules. First, we assume octave equivalence. This is by no means an odd or unintuitive exercise—but it necessarily excludes scales that do not repeat at the octave. I leave the question of non-octave-repeating FiPS open for now. Second, we assume symmetry in our 'chromatic' output universe.[12] In the common $7 \rightarrow 12$ configuration, for example, we arrange our 12 chromatic notes equally around the octave. Thus our J function finds the maximally even arrangement of seven notes *specifically in 12-tone equal temperament*. But we have the whole circle—geometrically, we could easily model alternate tuning systems by moving around points on the outermost circle, and our results would be significantly different! But the J-function does not allow this. Already we see a break between our geometric and algebraic models. And this reveals the third and most limiting assumption: that maximal evenness is an important property of maximally even sets.

An example to show why this matters: we can define the diatonic collection as the maximally even set of notes in 12-tone equal-tempered pitch-class space. But there are infinitely many ways to define the diatonic collection in terms of *quantization* that do not rely on the J-function's definition of maximal evenness! We could quantize seven equal-spaced notes to twelve-note-space-with-a-pentatonic-collection- removed; we could move one of our inner seven points some small amount $0 < \varepsilon < 1/84$ clockwise and rest assured that nothing changes in the output; we could try a $7 \rightarrow 7 \rightarrow 12$ or an $8 \rightarrow 7 \rightarrow 12$ or any $x \rightarrow 7 \rightarrow 12$ where $x > 7$ (and none of these necessarily evenly spaced!) and still change nothing. Except—once we set the circles in motion we won't necessarily

[12] In order to get a first-order maximally even set out of any given (though not a specific) output configuration, our input must be perfectly even. This is why I make the distinction between input and output.

get a progression by circle of fifths, but rather some other sequence that derives <u>not</u> from any inherent properties of the diatonic collection, but simply from the geometry that we consciously built in order to represent it!

Fig. 2: five different ways to generate the Db diatonic collection



The point here is that FiPS, under its current constraints, conflates intrinsic and extrinsic properties of the sets that it is confined to generate. Maximal evenness is one property that certain collections *have*, not a definition of what they *are*. And the quantization of maximally even sets to one another is a specific form of the general act of quantizing any set to any other set. Here we get a taste of Tymoczko's frustration: the J-function isn't a *definition* of FiPS; it's the S constraint on FiP! This very constraint forces us to add inexplicable layers in order to engineer desired outputs, unnecessarily complicates models of common-yet-uneven collections, limits the generality of our geometries, clutters up our conceptual understanding of quantization. By doing away with the need for symmetry, we can construct a far more robust theory of contextualization via quantization, a theory that has immediate analytical potential for a wide variety of musics.

III. Fun with Rectangles

So what is FiP without S? Having done away with the J function, we have to define a new, analogous function that drives the engine of the filtered points. It turns out to be quite simple: given a set

P of points p, each with a given radius r and phase θ (formally: $P \subseteq \{r \in N,\ \theta \in [0, 1)\}$), we say point $p \in P$ is **active** iff:

a) $r(p) = 1$, or

b) For some active q, $p \in P_0 = \{p_0 \mid r(p_0) = r(q) + 1\}$ and

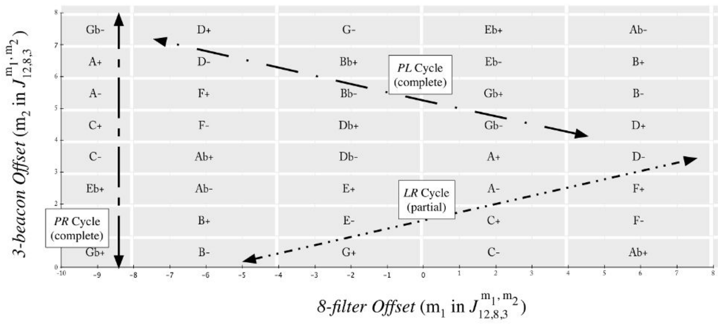
$(\theta(p) - \theta(q))\%1 < (\theta(p_0) - \theta(q))\%1\ \forall p_0 \neq p$

Which is to say: all points at radius 1 are active; and for any active point the nearest point counterclockwise on the next radius is also active. The astute reader will note that every configuration possible through the J-function is also possible with this algorithm. This one also allows configurations that are not symmetric. As before, to rotate a circle we just increment the phase of all points of a certain radius by some value *m*.

Of course, in doing this we have greatly expanded the set of possible harmonies configurable through FiP. And while one could—and should—spend hours graphically realizing various configurations instead of, say, making timely progress on one's term paper, this approach ultimately does not help predict the possible harmonies achievable through the various rotations of a given configuration. To solve this issue, we can import the concept of a configuration space—a map of all the possible resultant harmonies given the rotations of some amount of circles in a configuration (one axis per circle), assuming all the other circles are still at some known configuration. Reproduced below from Plotkin (2019) is a $3 \rightarrow 8 \rightarrow 12$ FiPS configuration space focusing on the two inner circles.[13] An important property of these spaces is that the set of paths possible along a space are exactly the progressions possible as a result of some continuous rotation of the circles represented—and any straight lines represent progressions that result from rotation at some constant speed.

[13] Plotkin (2019), "Chord Proximity, Parsimony, and Analysis with Filtered Point-Symmetry" *Music Theory Online* Vol. 25.2

Fig. 3: $3 \rightarrow 8 \rightarrow 12$ *FiPS configuration with Neo-Riemannian cycles shown as paths.*



Small changes in configuration can lead to incredible differences in configuration space—a fact worth exploring in detail in a much longer paper.[14] Let us derive one such alternative space, and let us derive it in a manner that holds an intuitive analytic value. The $3 \rightarrow 7 \rightarrow 12$ symmetric configuration makes intuitive sense as a means of understanding triads in a diatonic context, and that diatonic context in a chromatic context via modulation along the circle of fifths. Plotkin (2019) explores a $3 \rightarrow 7 \rightarrow 7 \rightarrow 12$ configuration, where the doubling of the 7-filters allows us to transpose our triads by diatonic steps.[15] The configuration space of the innermost two circles is reproduced below:

Fig. 4: $3 \rightarrow 7 \rightarrow 7 \rightarrow 12$ configuration space

3-filter Offset (m_3 in $J^{5,m_2,m_3}_{12,7,7,3}$)

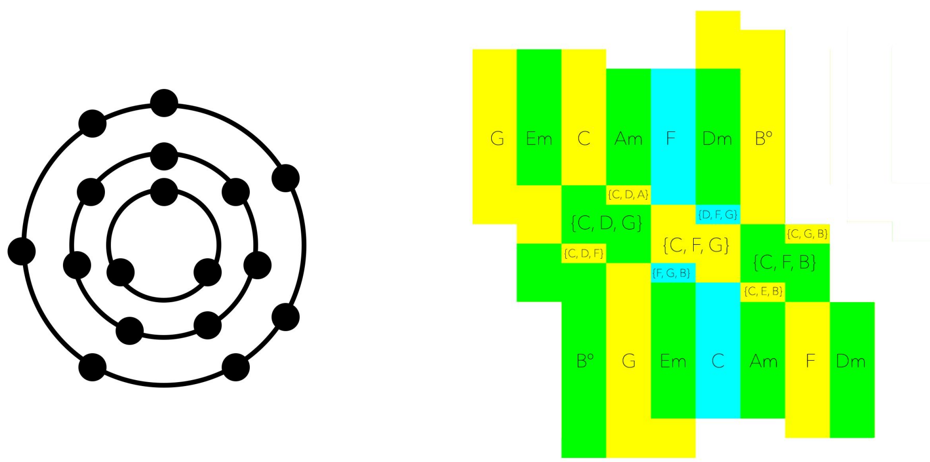
(C, +)	(D, -)	(E, -)	(F, +)	(G, +)	(A, -)	(B, °)	**(C, +)**	(D, -)
(E, -)	(F, +)	(G, +)	(A, -)	(B, °)	**(C, +)**	(D, -)	(E, -)	(F, +)
(G, +)	(A, -)	(B, °)	**(C, +)**	(D, -)	(E, -)	(F, +)	(G, +)	(A, -)
(B, °)	**(C, +)**	(D, -)	(E, -)	(F, +)	(G, +)	(A, -)	(B, °)	**(C, +)**
(D, -)	(E, -)	(F, +)	(G, +)	(A, -)	(B, °)	**(C, +)**	(D, -)	(E, -)
(F, +)	(G, +)	(A, -)	(B, °)	**(C, +)**	(D, -)	(E, -)	(F, +)	(G, +)
(A, -)	(B, °)	**(C, +)**	(D, -)	(E, -)	(F, +)	(G, +)	(A, -)	(B, °)
(C, +)	(D, -)	(E, -)	(F, +)	(G, +)	(A, -)	(B, °)	**(C, +)**	(D, -)

Inner 7-filter Offset (m_2 in $J^{5,m_2,m_3}_{12,7,7,3}$), moving in increments of +7

[14] Let's talk after the break.
[15] Plotkin (2019), "Chord Proximity, Parsimony, and Analysis with Filtered Point-Symmetry" *Music Theory Online* Vol. 25.2

But suppose we had the inner 7-filter output to a 7[*diatonic*] filter—that is, the actual positions in pitch-class space of the notes in the diatonic collection, set arbitrarily to C major.[16] The configuration and a section of the configuration space are shown below.

Fig. 5: 3 → 7 → 7[*C diatonic*] *configuration and section of configuration space. The 3-filter spins clockwise with respect to the 7-filter as we travel right along the configuration space; the 7-filter spins clockwise with respect to the 7[diatonic] filter as we go downwards.*



The space hints at the conceptual payload for this altered configuration: while we retain the third-relations and scale-step relations between chords as seen in Plotkin's symmetric configuration, the slight irregularities between our perfectly even and maximally even concepts of diatonicism uncover some intermediate spaces between the regular diatonic triads—that is, we have uncovered a space for suspensions and other irregular trichords. In doing so we have ruptured some of our previous connections between triads—it seems that relationships by third are robust under asymmetric conditions, but other relationships are not. The rules of voice-leading are very clear here: lateral and vertical motion

[16] This is analogous—but not equivalent—to collapsing together the outermost two circles in Plotkin's configuration. Another possibility might be configuring the center circle in {0, 4, 7} positions, or doing away with the perfectly even 7-filter entirely. These would all produce different variants of the same general idea: triads moving in a diatonic environment. We're simply playing with how generally or specifically we would like to define a triad and a diatonic environment.

corresponds to a voice-leading by one scale step or third (the latter only in certain horizontal moves); diagonal motion corresponds to voice-leading by two scale steps or thirds. In short: we retain the power and logic of the symmetric $3 \to 7 \to 7 \to 12$, but situate it in a more diverse field of possible harmonies, one better attuned to the continuous motion that underlies our understanding of voice-leading distance. And it's three-colorable!

Further study of asymmetrical configurations is sure to yield more powerful and perhaps even simpler results than this particular exercise. For the sake of brevity I will limit my explorations in this paper to the altered $3 \to 7 \to 7[\textit{diatonic}]$ configuration as described above. I expect, upon the completion of satisfactory exploration into this topic, to put together a more comprehensive *general* theory of FiP, or at least to describe a few more of its conceptually potent applications.

IV. Case Study: Chopin Op. 28 No. 9

To test the analytical value of our expanded FiP machine—beyond pure speculation—let us take a look at the last four measures of Chopin's Prelude Op. 28 No. 9 in E major, reproduced from Plotkin (2019) below:[17]

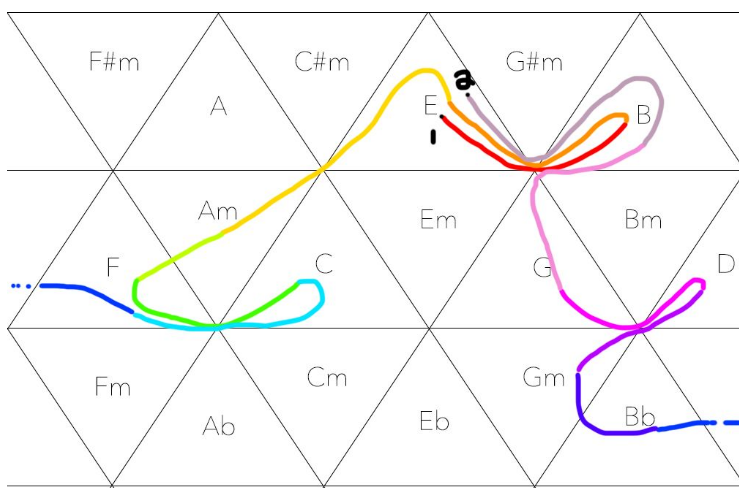
Fig. 6: Last four measures, Chopin Prelude Op. 28 No. 9 in E major

[17] Plotkin (2019), "Chord Proximity, Parsimony, and Analysis with Filtered Point-Symmetry" *Music Theory Online* Vol. 25.2

We start with a I-V-I in E major, and then a tonally ambiguous Am chord leads to a I-V-I in F major, a semitone up. Then something similar happens: a Bb chord leads to a G minor chord, two semitones up from F major. Then the sequence breaks: a i-V-I on G is followed by a B major—which is immediately recontextualized as a V of E major, and then the passage ends.

What is the appropriate space for such a progression? It seems to fit very elegantly in a simple Tonnetz, for one. And since the Tonnetz is isomorphic to a $3 \rightarrow 8 \rightarrow 12$ symmetric configuration space, the progression is elegantly traced out there as well.[18] Both are shown below:

Fig. 7: Chopin's quasi-sequential progression on a Tonnetz, starting at point 1 and changing color with each new chord through point 2. Note the self-similar movements through intersections



The implication here is that one may hear this passage as a series of P, L, and R transformations and their various combinations. While that's possible, this interpretation sacrifices notions of traditional functional harmony, i.e the dominant function of the chords B, C and D—even though Chopin all but hammers in a sense of functionality with his I-V-I progressions.

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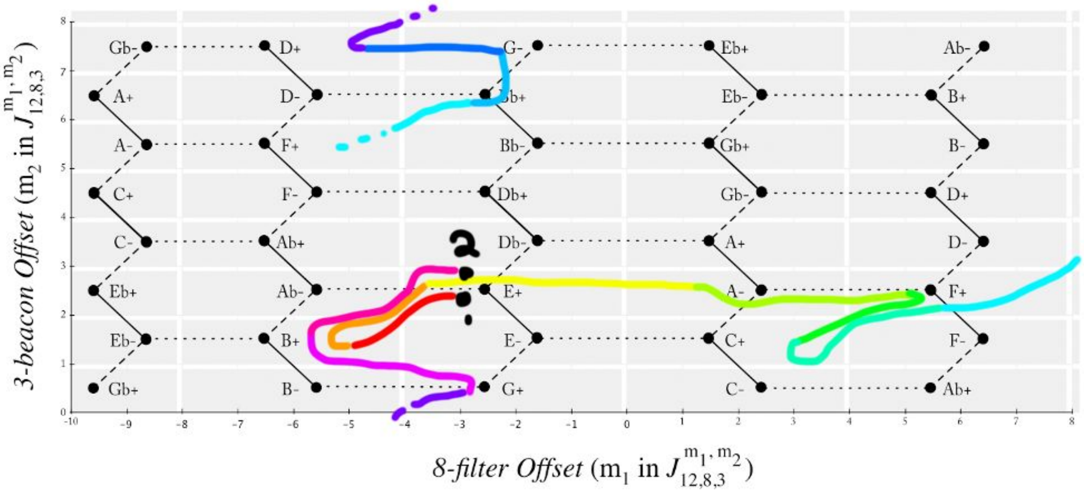
[18] *ibid.*

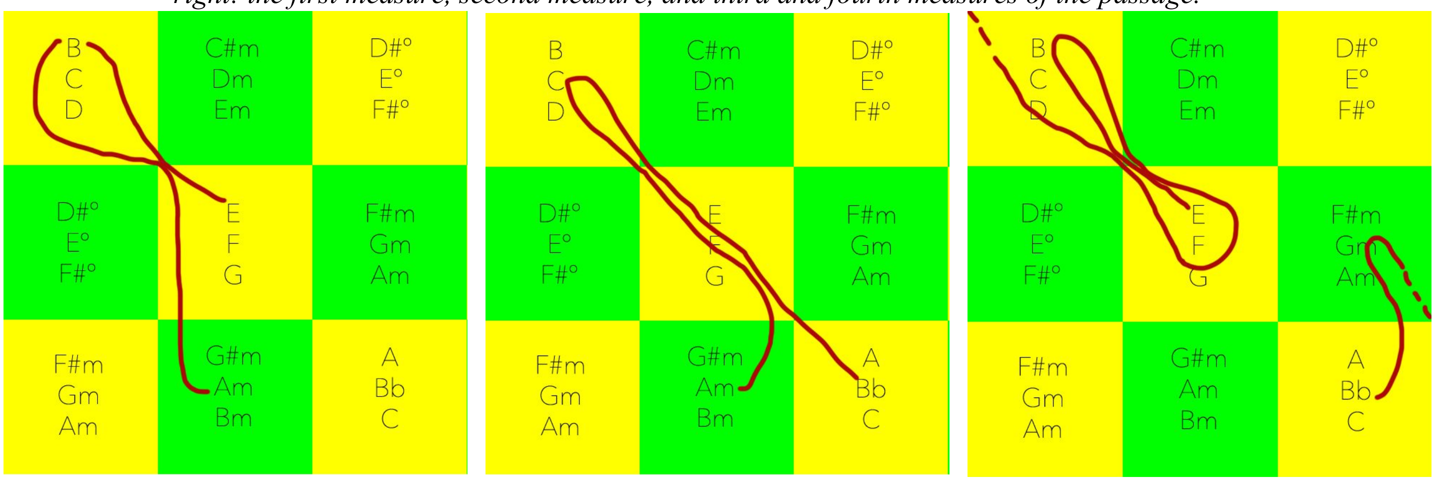
Fig. 8: The same passage on the $3 \to 8 \to 12$ *FiPS configuration space*



Plotkin (2019) suggests an alternative reading of the piece: in a $3 \to 7 \to 7 \to 12$ configuration, we can constrain ourselves to the triads in one diatonic collection—the whole of which we can then rotate through a second 12-filter in order to transpose upwards by semitones. This amounts to a monstrous $3 \to 7 \to 7 \to 12 \to 12$ configuration. The mechanism is clunky, but the conceptual basis is salient: We are looking at triads in a diatonic space; and we are progressively transposing that diatonic space upwards. In order to visualize this, we can imagine the familiar $3 \to 7 \to 7 \to 12$ configuration space, focusing on the innermost two circles, set to E major. We then overlay the same space set to F major and G major—in essence, squashing and twisting a three-dimensional space into a two-dimensional plane. We can think of this as a chord-function space that is movable through chromatic transpositions, with I in the middle, V to the top right, IV to the bottom left, and so on. I have mapped the progression on that space, separating the progression by measure (with the last 2 together) in order to avoid clutter.

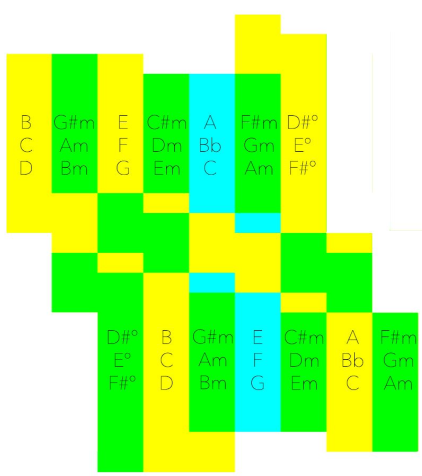
Immediately we note the same loop pattern we saw in the Tonnetz, now confined to the top right corner. These are the I-V-I progressions. We no longer hear modulatory pivot chords as tonally ambiguous PLR transformations, however—simply as functional chords in the new diatonic transposition. Interestingly, we do not enter the key of G minor here (recall that FiPS has trouble with non-ME sets like minor scales with a sharp seventh scale degree) but rather G Dorian. So we still sacrifice a lot of information about how one may hear this passage; and we've gone through a lot of math to come to an analysis that is musically rather straightforward—simply that the passage could be heard entirely diatonically, with no chromatic ambiguity.

Fig. 9: Chopin's progression in $3 \rightarrow 7 \rightarrow 7 \rightarrow 12 \rightarrow 12$ *FiPS configuration space. From left to right: the first measure, second measure, and third and fourth measures of the passage.*



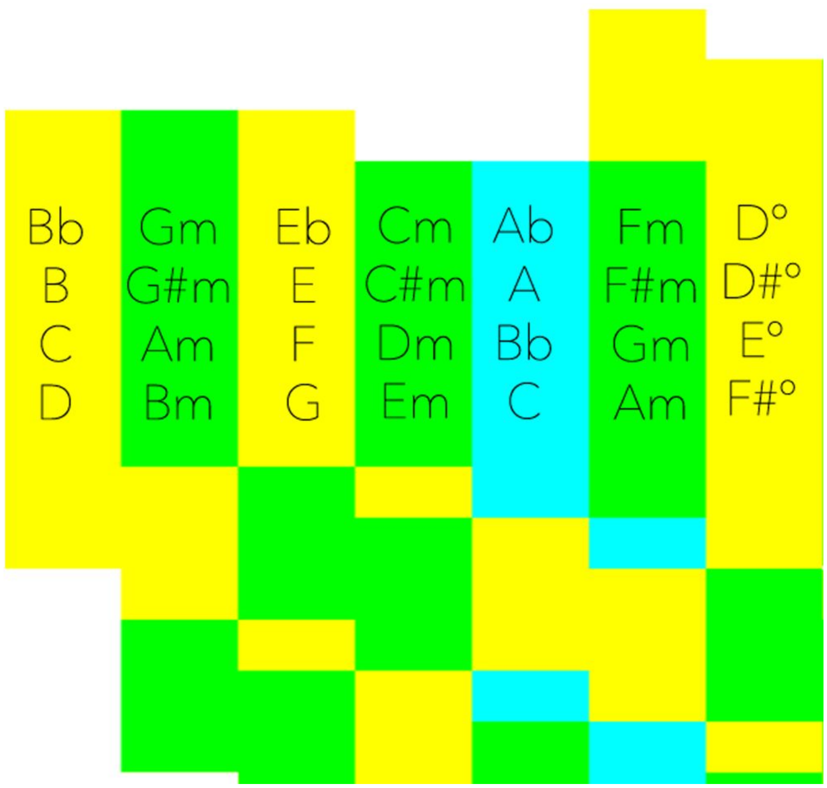
Finally, let us test out our engineered $3 \rightarrow 7 \rightarrow 7[\textit{diatonic}]$ configuration. In order to move from key to key, we have to situate it inside of a 12-filter. This allows us to transpose everything by semitones, just like in the $3 \rightarrow 7 \rightarrow 7 \rightarrow 12 \rightarrow 12$. Let us enact the same mental squashing exercise to arrive at the following compound configuration space:

Fig. 10: $3 \rightarrow 7 \rightarrow 7[\textit{diatonic}] \rightarrow 12$ *FiP configuration space, collapsed to show multiple transpositions. Non-triadic trichords not notated.*



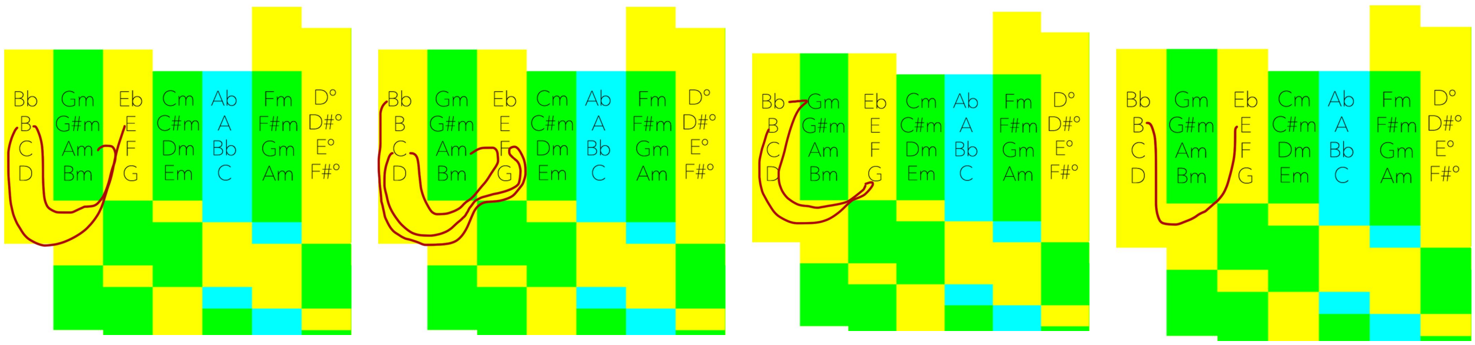
Immediately we see a problem: there is no path leading from a I chord to a IV chord! When we reach the fourth beat of the second measure, we can't get from F to Bb. And when we get to Gm, we can't reach D. Has FiP failed us so soon? Not quite—recall that this space squashes down *every* transposition of a diatonic collection into this space. Figure 10 only shows 3 of the 12 chords inhabiting each region. We can solve our problem by simply considering one more possible diatonic collection: Eb major.

Fig. 11: $3 \rightarrow 7 \rightarrow 7[\textit{diatonic}] \rightarrow 12$ *FiP configuration space, showing Eb, E, F and G diatonic collections.*



This solves our problem quite nicely, as we can chart the following set of paths:

Fig. 12: Chopin's progression in $3 \rightarrow 7 \rightarrow 7[\textit{diatonic}] \rightarrow 12$ *FiP configuration space. From left to right: the first, second, third, and fourth measure of the passage.*



And now we've uncovered a far less obvious interpretation of the passage! We restrict ourselves into looking at every chord as a I/i, iii, or V of a diatonic collection. In doing so we situate the Bb to G minor progression as a deceptive cadence in Eb major. Now we have doubly highlighted the sequential or iterative structure of the passage—first by introducing extreme regularity in our motion through our chosen space, moving back and forth between the same two regions all but twice; second by suggesting a tonal gravity around Eb during the first beat of the third measure. We transpose our diatonic collection *up* a semitone from E to F, and then *down* two semitones into Eb before ricocheting up four semitones into G. Our sequence of roots is not *ascending* but *oscillating* around E—hearing it that way makes that final V-I cadence into E not a deviation from the sequence, but its logical conclusion: the spring settles into place at equilibrium.

V. Conclusion

Our asymmetrical FiP configuration allows us to uncover a much more nuanced yet equally valid possible hearing of Chopin's prelude as compared to the alternatives posed by the Tonnetz and FiPS. And the linkage of conceptual rigor to a highly specific geometry allows us to interpret the challenges and workarounds of its topology in an analytically productive way. Therein lies the potential of FiP: the ability to generate the possibility space for incredibly specific musical situations in a relatively simple way, without the need to jump through the hoops of iterated symmetry.[19]

A more important point: this case study supports my notion that there is no unifying property of 18th and 19th-century tonality. Maximal evenness, neo-Riemannian transformations, and voice-leading efficiency are tools that we can use to understand and hear certain progressions, but the context of a passage can endlessly and ambiguously redefine the relationship between one harmony and the next. By taking maximal evenness out of the equation in FiPS, I don't necessarily intend to create the universal computer of harmonies through whose various configurations we may gain infinite wisdom. I simply hope to present it as a more powerful tool for addressing music-analytic problems in less-than-obvious ways.

[19] Pun intended.